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The development of three-dimensional wave packets in a boundary layer

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The formation and growth of three-dimensional wave packets in a laminar boundary layer is treated as a linear problem. The asymptotic form of the disturbed region developing from a point source is obtained in terms of parameters describing two-dimensional instabilities of the flow. It is shown that a wave caustic forms and limits the lateral spread of growing disturbances whenever the Reynolds number is $\surd 2$ times the critical value. The analysis is applied to the boundary layer on a flat plate and shapes of the wave-envelope are calculated for various Reynolds numbers. These show that all growing disturbances are contained within a wedge-shaped region of approximately 10° semi-angle.

1. Introduction

The linear theory of hydrodynamic stability examines the possible perturbations to a given mean flow and determines the various eigenvalues defining these disturbances. In boundary layers the eigenmodes take the form of travelling waves. Disturbances may be generated by perturbing one or more of the boundary conditions. A flat ribbon oscillating vertically close to the surface is one example where the resulting downstream motion is a single two-dimensional wave. Such a disturbance was used by Schubauer & Skramstad (1947) to excite the boundary layer on a flat plate so that the theoretical predictions of Schlichting (1933, 1935) could be verified. Other boundary perturbations generate more complicated flows which are composed of linear combinations of simple waves. In this paper the boundary perturbation has the form of a pulsed point source and the resulting disturbance is a three-dimensional wave packet. Apart from any intrinsic interest in the behaviour of such a wave packet, the solution may be more applicable to natural transition than the single wave model generated by a vibrating ribbon. Natural excitation can probably be better simulated by a series of pulse perturbations distributed randomly in space and time. Far downstream the patches of waves coalesce and form the observed type of irregular oscillation.

The asymptotic form of the wave packet developing from a point source has been studied by Benjamin (1961) and by Criminale & Kovasznay (1962). In both cases approximations were made to simplify the integration, and details of the wave envelope shape were lost. In this paper, which also uses asymptotic expansions, different approximations will be made so that a more complete solution of the problem is obtained. The analysis applies specifically to those boundary-layer flows which admit waves having small rates of amplification.

Although the problem to be treated is three-dimensional it is convenient to discuss first the simpler two-dimensional case. The same technique will then be applied to the general three-dimensional problem, making use of two-dimensional data through Squire's (1933) transformation. The resulting relationships are applied to the boundary layer on a flat plate and the shape and growth of the wave packet is obtained for a range of Reynolds numbers.

2. Formulation

The mean flow is treated as being parallel and the equations governing the perturbation are linearized. These homogeneous equations reduce to the Orr–Sommerfeld equation in the transform plane. The usual homogeneous boundary conditions yield the familiar eigenvalue problem of stability theory, forced motions being generated by inhomogeneities in one of the boundary values. The analysis presented here follows that used in the solution of the vibrating Schubauer ribbon (Gaster 1965), with similar assumptions and approximations. The present problem differs from that previously discussed in the choice of boundary conditions. However, it is convenient to use the results already obtained to derive the solution to the present problem. This removes the need to repeat all the arguments concerned with paths of integration and convergence of the transform integrals etc.

The ribbon, vibrating at frequency ω, was represented by the following boundary conditions on the perturbation streamfunction: at the wall

$$y = 0, \quad u = \partial\psi/\partial y = 0, \quad v = -\partial\psi/\partial x = \delta(x)\cos\omega t H(t)$$

(where $H(t)$ is the Heaviside operator); far from the wall at y_1 the perturbation decays, i.e. as $y_1 \to \infty$

$$\partial\psi/\partial y \to 0 \quad \text{and} \quad \partial\psi/\partial x \to 0.$$

The asymptotic solution for the usual case when the group velocity of the excited mode is positive was shown to be

$$v(y;x,t) = \mathscr{R}\left\{-\frac{i}{2}\frac{\Phi(y;\alpha(\omega),\omega)}{\partial\Phi(0;\alpha(\omega),\omega)/\partial\alpha}\exp\{i(\alpha(\omega)x-\omega t)\}H(x)\right\}, \tag{1}$$

where $\mathscr{R}\{\}$ is the real part of $\{\}$, and Φ is the transform of the perturbation streamfunction $\psi(y;x,t)$.

The eigenvalue equation for the normal modes of the system is given by $\Phi(0;\alpha(\omega),\omega) = 0$. The wave-number, α, takes the appropriate value, which is generally complex, for this relation to hold. The travelling wave described by (1) is the spatially growing disturbance which is observed downstream of a two-dimensional Schubauer ribbon.

A two-dimensional pulse input may be represented by a boundary perturbation of the form:

$$\frac{\partial\psi}{\partial y} = 0, \quad -\frac{\partial\psi}{\partial x} = \delta(x)\delta(t),$$

where δ is the unit impulse function. Now the delta function may be written in the form of an integral

$$\delta(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \cos \omega t H(t)\, d\omega,$$

and we can therefore obtain the asymptotic form of the streamfunction for the pulse disturbance as an integral,

$$v = \mathscr{R}\left\{\int_{-\infty}^{+\infty} \frac{-i\Phi(y;\alpha(\omega),\omega)\, H(x)}{\partial\Phi(0;\alpha(\omega),\omega)/\partial\alpha} \exp\{i[\alpha(\omega)x-\omega t]\}\, d\omega\right\}.$$

In general it is found convenient to evaluate integrals of this type in the complex plane. All the functions can be defined in the complex domain by replacing ω by the more general complex parameter β.

3. The asymptotic expansion

It is required to evaluate an integral of the form:

$$I = \int_{-\infty}^{+\infty} \lambda(\alpha,\beta)\exp\{i(\alpha x-\beta t)\}\, d\beta,$$

where $\lambda(\alpha,\beta)$ is a regular function of the complex parameters α and β. If $\bar{\lambda}(\alpha,t)$ is the Fourier transform of λ, and $P(\alpha,t)$ the transform of $\exp(i\alpha x)$ we may use the relation for the transform of a product:

$$I = \int_{-\infty}^{+\infty} \bar{\lambda}(\alpha, t-\eta)\, P(\alpha,\eta)\, d\eta.$$

Both of the terms in the integrand can focus attention on regions of the spectrum which give large contributions to the integral. However, for large t the dominant contribution arises from the exponential function and, as in the previous studies of Benjamin and Criminale & Kovasznay, only the behaviour of this term will be considered. The problem thus reduces to finding the asymptotic expansion of

$$I \sim \int_{-\infty}^{+\infty} \exp\left\{i\left(\alpha\frac{x}{t}-\beta\right)t\right\} d\beta \text{ for large } t.$$

x is also a large parameter and x/t is of order unity.

Asymptotic expansions of integrals of this form can be obtained by applying the method of steepest descent. The contour is deformed to pass through the saddle-point of the exponent in such a way that the periodic part of the integrand is constant and the large exponential term is at a maximum. The largest values of the integrand arise in the region of the saddle-point and the first term of the asymptotic series, which is sufficient for the present discussion, can be obtained in terms of parameters evaluated at the saddle-point.

The exponent is stationary at the point denoted by β^* where

$$\frac{d\alpha}{d\beta}(\beta^*) - \frac{t}{x} = 0,$$

or in terms of real and imaginary parts

$$\left.\begin{aligned} \frac{t}{x} &= \frac{\partial \alpha_r}{\partial \beta_r}(\beta^*) \\ \text{and} \qquad 0 &= \frac{\partial \alpha_i}{\partial \beta_r}(\beta^*). \end{aligned}\right\} \qquad (2)$$

These two equations define the complex quantity β^* in terms of t/x. Different rays in the (x, t)-plane lead to different values of β^*.

We may expand the exponent about β^*

$$i\left[\alpha\frac{x}{t} - \beta\right] = i\left[\alpha(\beta^*)\frac{x}{t} - \beta^*\right] - i\frac{(\beta-\beta^*)^2}{2}\frac{x}{t}\frac{d^2\alpha}{d\beta^2}(\beta^*) + \dots. \qquad (3)$$

As the path is chosen to pass through the saddle-point so that the imaginary part of the exponent remains constant we have that

$$i\left[\frac{(\beta-\beta^*)^2}{2}\frac{d^2\alpha}{d\beta^2}(\beta^*) + \dots\right] \text{ is purely real.}$$

The first term of the expansion thus becomes

$$\sim \frac{\exp\{i[\alpha(\beta^*)x - \beta^* t]\}}{d^2\alpha(\beta^*)/d\beta^2}, \qquad (4)$$

neglecting a phase factor which depends on the direction of the path of integration. Equation (4) represents a travelling wave system which is amplified in both space and time. The values of the real and imaginary parts of both wave-number and frequency vary slowly in space and time through the position of the saddle-point defined by (2). Along rays in the (x, t)-plane, β^* is constant and (4) appears as a simple travelling wave with wave-number $\alpha_r(\beta^*)$, frequency β_r^* and amplification

$$-[\alpha_i(\beta^*)x - \beta_i^* t]. \qquad (5)$$

The amplification rate given by (5) may be written in terms of temporal growth

$$-\left[\alpha_i(\beta^*)\frac{\partial \beta_r}{\partial \alpha_r}(\beta^*) - \beta_i^*\right]t,$$

or spatial growth

$$-\left[\alpha_i(\beta^*) - \beta_i^*\frac{\partial \alpha_r}{\partial \beta_r}(\beta^*)\right]x.$$

Regions of amplified waves exist in the physical (x, t)-plane where (5) is positive. The amplified region is thus bounded by the rays which make (5) zero. The leading and trailing edges move with the speeds of the group velocities appropriate to the waves on the neutral rays, and thus the pulse spreads out linearly with time (or space) as it propagates downstream. Except for a qualitative description of the behaviour the relations given by (2) and (4) are not very convenient for calculating the actual behaviour of the wave packet. Eigenvalues for temporally growing waves are usually the only ones computed and for the present purpose we need to know the behaviour of α and $\partial\alpha_r/\partial\beta_r$ along a different path on the β-plane.

However, if one makes certain approximations it is possible to obtain the necessary quantities in (2) and (4) solely from a knowledge of the temporally growing modes.

4. Approximation for small amplification rates

Instability waves in flat plate boundary layers have low amplification rates. Calculated values of the temporal amplification rate β_i for zero spatial growth ($\alpha_i = 0$) are quite small compared with the real parameters α_r and β_r. We can also expect $\partial\alpha_i/\partial\beta_r$ to be small along the line $\alpha_i = 0$ in the β-plane. Now $\partial\alpha_i(\beta^*)/\partial\beta_r$ is zero and since this derivative is already small we can expect the saddle-point to lie close to the $\alpha_i = 0$ contour. With the assumption that α is an analytic function of β in this region the Cauchy–Riemann relations hold,

$$\frac{\partial\alpha_r}{\partial\beta_r} = \frac{\partial\alpha_i}{\partial\beta_i} \quad \text{and} \quad \frac{\partial\alpha_r}{\partial\beta_i} = -\frac{\partial\alpha_i}{\partial\beta_r}.$$

Differentiating we also obtain

$$\frac{\partial^2\alpha_r}{\partial\beta_r^2} = \frac{\partial^2\alpha_i}{\partial\beta_r\partial\beta_i}, \text{ etc.}$$

If we integrate the above with respect to β_i from state (1) to state (2) keeping β_r constant we have

$$\left[\frac{\partial\alpha_i}{\partial\beta_r}\right]_{(1)}^{(2)} = \int_{(1)}^{(2)} \frac{\partial^2\alpha_r}{\partial\beta_r^2} d\beta_i. \tag{6}$$

Let state (1) be a known temporally growing mode and state (2) the saddle-point, then

$$\alpha_i(1) = 0, \quad \frac{\partial\alpha_i}{\partial\beta_r}(2) = 0 \quad \text{and} \quad \beta_r(1) = \text{constant} = \beta_r(2).$$

It can then be shown by expanding the integrand and integrating (6) (see Gaster 1962) that

$$-\frac{\partial\alpha_i(1)}{\partial\beta_r} = \frac{\partial^2\alpha_r(1)}{\partial\beta_r^2}[\beta_i(2)-\beta_i(1)] + O(\beta_{im}^3). \tag{7}$$

Thus $[\beta_i(2)-\beta_i(1)]$ is small and the saddle-point lies close to a temporally growing mode provided $\partial^2\alpha_r(1)/\partial\beta_r^2$ is not small. Using similar techniques we can obtain the values of all the saddle-point parameters arising in (4) in terms of quantities for temporally growing modes. The Cauchy–Riemann relations can be integrated at constant β_r from (1) to (2) (see Gaster 1962), giving to order (β_{im}^2):

$$\alpha_r(2) = \alpha_r(1), \quad \frac{\partial\alpha_r}{\partial\beta_r}(2) = \frac{\partial\alpha_r}{\partial\beta_r}(1), \text{ etc.}$$

and

$$\alpha_i(2) = [\beta_i(2)-\beta_i(1)]\frac{\partial\alpha_r}{\partial\beta_r}(1).$$

The predominant term in the asymptotic series thus reduces to

$$\frac{\exp\{i[\alpha_r(1)\partial\beta_r(1)/\partial\alpha_r - \beta_r(1)]t\}\exp\{\beta_i(1)t\}}{[\{\partial^2\alpha_r(1)/\partial\beta_r^2\}x]^{\frac{1}{2}}}. \tag{8}$$

For large t the wave envelope is contained within the rays of zero growth associated with the neutral two-dimensional eigenmodes. All temporally growing disturbances between these limits can be linked with a region inside the wave packet where the waves have equal values of wave-number, frequency and amplification rate.

5. A three-dimensional pulse

It is a relatively straightforward process to extend the two-dimensional solution to three dimensions. It is convenient to change notation at this stage and we will define a general three-dimensional wave in the form

$$v(y; x, z, t) = \phi(y) \exp\{i(ax + bz - \omega t)\},$$

where a and b are the wave-numbers in the x and z directions. In general a, b and ω will be complex.

Consider the boundary perturbation:

$$y = 0; \quad u = 0, \quad w = 0, \quad v = \delta(x)\,\delta(t) \exp(ibz). \tag{9}$$

Keeping b constant we can use the result obtained in §3, which gives

$$v \sim \exp(ibz)\, \mathscr{R}\left\{\int_{-\infty}^{+\infty} \frac{i\Phi(y; a(\omega, b), b, \omega)}{\partial\Phi(0; a(\omega, b), b, \omega)/\partial a} H(x) \exp\{i[a(\omega, b)\, x - \omega t]\}\, d\omega\right\}.$$

The eigenvalues of the three-dimensional wave system are given by

$$\Phi(0;\, a(\omega, b), b, \omega) = 0;$$

note that the wave-number in the x-direction is now a function of b as well as ω. Concentrating on the exponential term we can expand as before,

$$I \sim \exp(ibz) \int_{-\infty}^{+\infty} \exp\left\{i\left[a(\omega, b)\frac{x}{t} - \omega\right] t\right\} d\omega.$$

Expanding the exponent about the saddle-point ω^* on the ω-plane we obtain

$$I \sim \frac{\exp(ibz) \exp\{i[a(\omega^*, b)\,(x/t) - \omega^*]t\}}{[\partial^2 a/(\omega^*, b)/\partial\omega^2 x]^{\frac{1}{2}}}, \tag{10}$$

where the saddle-point is given by the complex equation

$$\frac{\partial a}{\partial \omega}(\omega^*, b)\frac{x}{t} - 1 = 0.$$

Since the integration of (9) with respect to b leads to a perturbation of the form $\delta(x)\,\delta(z)\,\delta(t)$, we can arrive at the downstream disturbance by integrating (10),

$$I \sim \int_{-\infty}^{+\infty} \frac{\exp\{i[b(z/t) + a(\omega^*, b)\,(x/t) - \omega^*]\, t\}}{[\partial^2 a(\omega^*, b)/\partial\omega^2 x]^{\frac{1}{2}}}\, db.$$

Expanding the exponent about b^* we have

$$i\left[b^*\frac{z}{t} + a(\omega^*, b^*)\frac{x}{t} - \omega^*\right] t + i\left[\frac{z}{t} + \frac{\partial a}{\partial b}(\omega^*, b^*)\frac{x}{t}\right] t(b - b^*)$$
$$+ i\left[\frac{\partial^2 a}{\partial b^2}(\omega^*, b^*) + 2\frac{\partial^2 a}{\partial\omega\,\partial b}(\omega^*, b^*)\frac{d\omega^*}{db} + \frac{\partial^2 a}{\partial\omega^2}(\omega^*, b^*)\left(\frac{d\omega^*}{db}\right)^2\right]\frac{x(b - b^*)^2}{2!} + \ldots.$$

If we choose b^* to be the saddle-point we can make the linear term vanish

$$\frac{z}{t}+\frac{x}{t}\frac{\partial a}{\partial b}(\omega^*, b^*) = 0,$$

or

$$\frac{z}{t} = \frac{\partial \omega}{\partial b}(\omega^*, b^*).$$

I reduces to

$$\frac{\exp\{i[b^*z+a(\omega^*,b^*)x-\omega^* t]\}}{x\left[\frac{\partial^2 a}{\partial b^2}\frac{\partial^2 a}{\partial \omega^2}-\left(\frac{\partial^2 a}{\partial b\,\partial \omega}\right)^2\right]^{\frac{1}{2}}}, \tag{11}$$

where the derivatives are evaluated at (ω^*, b^*).

Now b^*, ω^* and $a(\omega^*, b^*)$ are complex quantities, but if we again assume that the imaginary parts are small compared with the real it is possible to evaluate all the terms in (11) for temporally growing modes which have a and b real, and ω complex. Such an expansion is only valid when a term equal to the denominator of (11) is not small. Treating ω as a complex function of the two real variables a and b we get the form:

$$v \sim \frac{\exp\{i[a^*x+b^*z-\omega(a^*,b^*)t]\}}{t\left[\frac{\partial^2 \omega_r}{\partial a^2}\frac{\partial^2 \omega_r}{\partial b^2}-\left(\frac{\partial^2 \omega_r}{\partial a\,\partial b}\right)^2\right]^{\frac{1}{2}}}, \tag{12}$$

where a^* and b^* are defined by the two equations

$$\frac{x}{t} = \frac{\partial \omega_r}{\partial a}(a^*, b^*) \quad \text{and} \quad \frac{z}{t} = \frac{\partial \omega_r}{\partial b}(a^*, b^*). \tag{13}$$

Thus for an x/t and z/t we can obtain ω_i, the amplification factor. The neutral loop, $\omega_i = 0$, on the (a, b)-plane transforms into the outline of the wave packet in the physical plane. On the inside of this boundary all waves are amplified and outside they are damped.

6. Squire's transformation

Squire (1933) applied a simple transformation to the perturbation equations of motion for a three-dimensional travelling wave and showed that the wave $\exp\{i[ax+bz-act]\}$ at Reynolds number R_0 was equivalent to the two-dimensional wave $\exp\{i[\alpha x-\alpha ct]\}$ at a lower Reynolds number R when

$$R/R_0 = a/\alpha \quad \text{and} \quad \alpha^2 = a^2+b^2; \tag{14}$$

α, a and b are taken to be real, while the wave speed, c, which is the same for both waves, is complex. The solution of the two-dimensional stability problem for temporally growing waves can be presented in the form $c = F_1(\alpha, R_0)$. This may be transformed by (14) into $c = F_2(a, b)$ for any Reynolds number R lower than R_0. Putting $r = R/R_0$ we get

$$r = \left[\frac{a^2}{a^2+b^2}\right]^{\frac{1}{2}}, \quad a = \alpha r \quad \text{and} \quad b = \alpha(1-r^2)^{\frac{1}{2}}. \tag{15}$$

The real wave-numbers a and b are functions of α and r so we can write

$$\left(\frac{\partial c_r}{\partial a}\right)_b = \left(\frac{\partial c_r}{\partial \alpha}\right)_r \left(\frac{\partial \alpha}{\partial a}\right)_b + \left(\frac{\partial c_r}{\partial r}\right)_\alpha \left(\frac{\partial r}{\partial a}\right)_b$$

and

$$\left(\frac{\partial c_r}{\partial b}\right)_a = \left(\frac{\partial c_r}{\partial \alpha}\right)_r \left(\frac{\partial \alpha}{\partial b}\right)_a + \left(\frac{\partial c_r}{\partial r}\right)_\alpha \left(\frac{\partial r}{\partial b}\right)_a.$$

But

$$\frac{\partial r}{\partial a} = \frac{1-r^2}{\alpha}, \quad \frac{\partial \alpha}{\partial a} = r,$$

$$\frac{\partial r}{\partial b} = -r\frac{(1-r^2)^{\frac{1}{2}}}{\alpha} \quad \text{and} \quad \frac{\partial \alpha}{\partial b} = (1-r^2)^{\frac{1}{2}},$$

so that

$$\left.\begin{aligned} \frac{\partial c_r}{\partial a} &= r\frac{\partial c_r}{\partial \alpha} + \left(\frac{1-r^2}{\alpha}\right)\frac{\partial c_r}{\partial r} \\ \text{and}\quad \frac{\partial c_r}{\partial b} &= (1-r^2)^{\frac{1}{2}}\left[\frac{\partial c_r}{\partial \alpha} - \frac{r}{\alpha}\frac{\partial c_r}{\partial r}\right]. \end{aligned}\right\} \tag{16}$$

Now $\omega = ac$; hence $\omega_r = \alpha r c_r$ and $\omega_i = \alpha r c_i$, and we have

$$\left.\begin{aligned} \frac{\partial \omega_r}{\partial a} &= c_r + \alpha r\frac{\partial c_r}{\partial a} \\ \text{and}\quad \frac{\partial \omega_r}{\partial b} &= \alpha r\frac{\partial c_r}{\partial b}. \end{aligned}\right\} \tag{17}$$

The relations given in (13), (16) and (17) provide the necessary link between the two-dimensional stability diagram and the wave envelope in the physical plane. Each point on the stability diagram contributes to the disturbance at some point on the $(x/t, z/t)$-plane. Regions inside the neutral loop contribute to areas of the wave envelope which have disturbances that increase with time, and neutral two-dimensional modes are linked with the boundary of the amplified region in the physical plane. Defining the outline of the wave envelope by this neutral amplification boundary enables the shape to be obtained simply in terms of the quantities c_r, $\partial c_r/\partial \alpha$ and $\partial c_r/\partial r$ evaluated on the neutral loop.

7. Numerical results

The familiar two-dimensional stability diagram for temporally growing waves in a Blasius flat plate boundary-layer profile is shown on figure 1. The numerical data for this plot was obtained by Dr M. R. Osborne at Edinburgh University. He also calculated the derivatives $\partial c_r/\partial \alpha$ and $\partial c_r/\partial R$, and these are plotted in figure 2 in a form which is convenient for the calculation of the wave packet shape. These curves have been used together with equations (13), (16) and (17) to find the wave-envelope boundary for a range of Reynolds numbers.

8. Discussion

At low Reynolds numbers the wave packet is crescent shaped. All amplified regions inside the neutral stability loop map onto the interior of this boundary. However, above some critical value of Reynolds number we find that different areas inside the neutral loop map onto the same region of the physical plane so that the neutral contour, which in fact crosses itself, cannot fully define the wave packet. This is the familiar wave caustic (see Lighthill 1965). The caustic arises when the transformation is singular and strips of width d on the stability diagram map onto regions of width d^2 on the physical plane. The waves pile up along a front and cause the motion to have a large disturbance amplitude there. The caustic occurs where the denominator of (11) is zero. In the region of the caustic, where the expansions used to derive (12) are invalid, the solution can be obtained

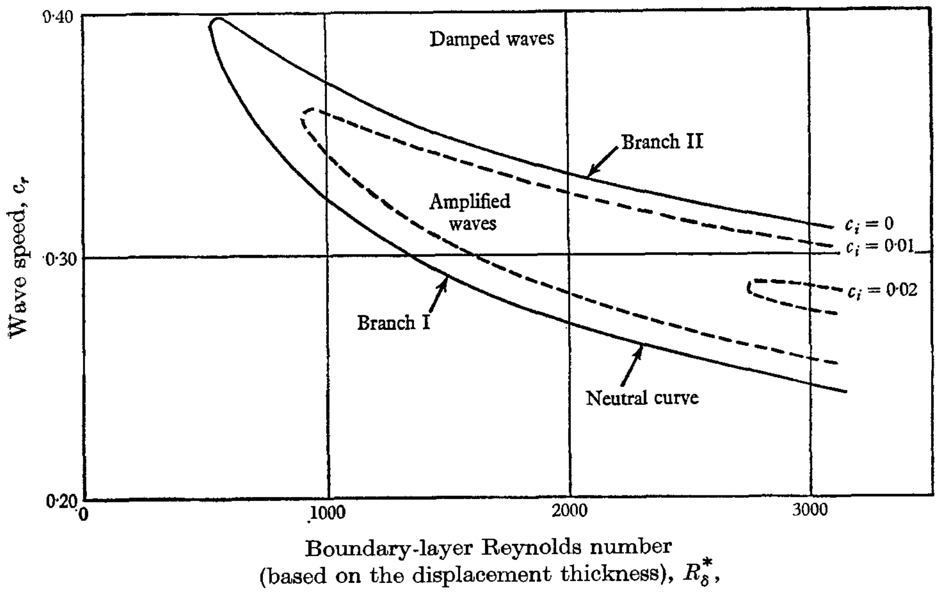


FIGURE 1. Amplification curves for two-dimensional waves in a zero pressure gradient boundary layer.

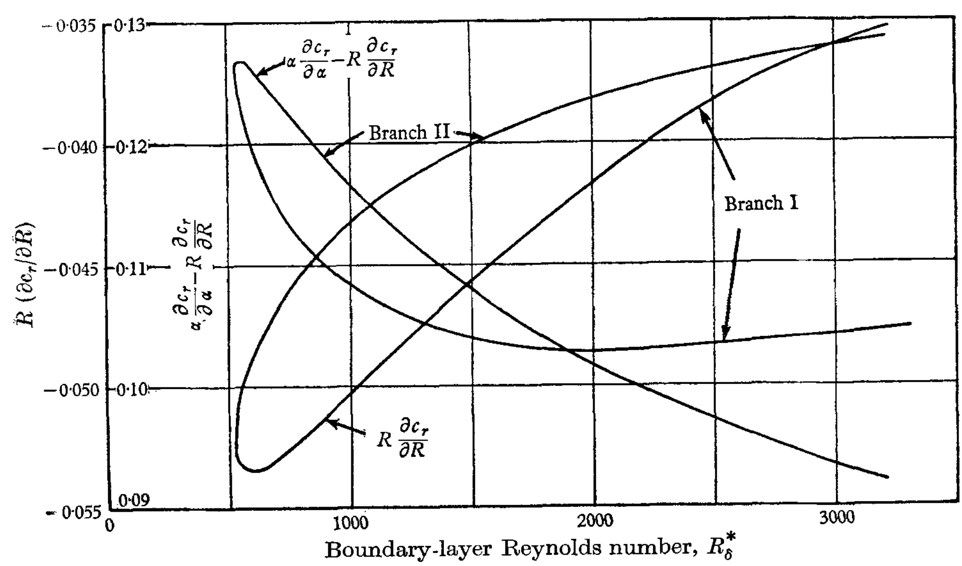


FIGURE 2. The behaviour of $R(\partial c_r/\partial R)$ and $\alpha(\partial c_r/\partial \alpha) - R(\partial c_r/\partial R)$ on the neutral curve.

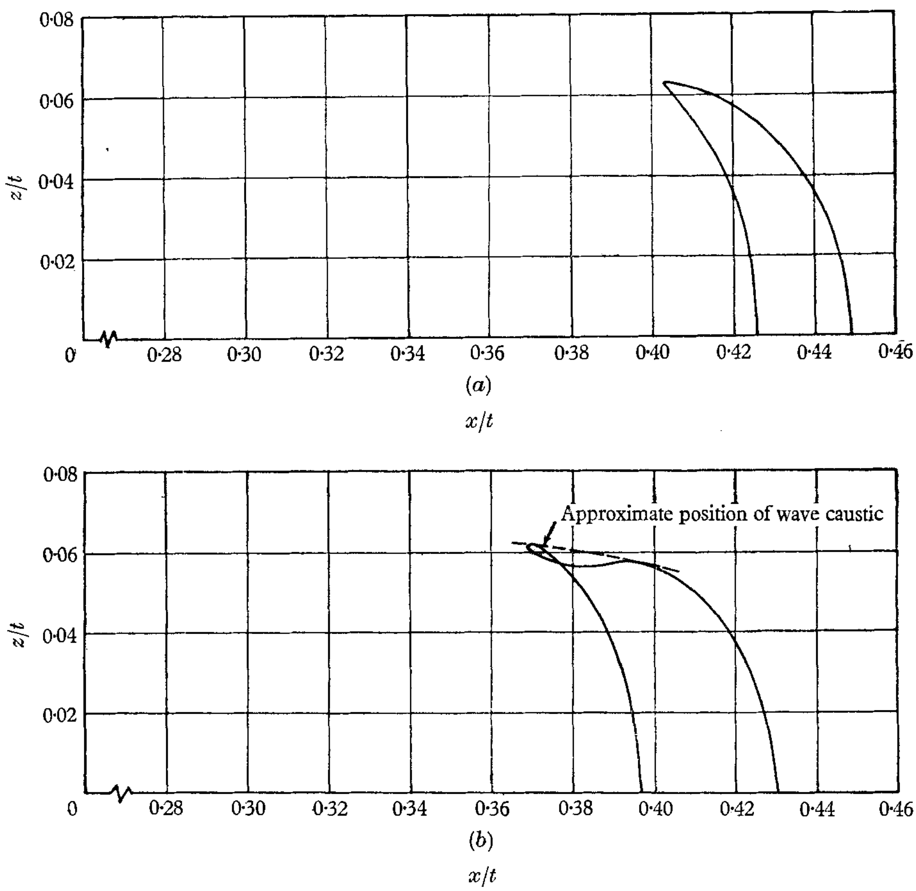
in the form of Airy functions from higher-order expansions. These solutions are periodic on one side of the caustic, reach a maximum amplitude on the caustic and decay exponentially away on the other side. On the caustic the disturbance, which has a larger amplitude than that given by (12), decays algebraically as $t^{-\frac{5}{6}}$ and increases exponentially at a rate determined by the region on the stability diagram which generates the caustic.

In the particular case of a Blasius flat plate profile we note that the value of $[\alpha(\partial c/\partial\alpha) - r(\partial c/\partial r)]$ does not vary a great deal around the neutral curve (see figure 2). The singularity in the transformation to the physical plane is therefore controlled by the term $r(1-r^2)^{\frac{1}{2}}$. This has a maximum at $r = 1/\sqrt{2}$ which leads to maximum values of

$$\frac{z}{t} \text{ of } \frac{1}{2}\left[\alpha\frac{\partial c}{\partial\alpha} - r\frac{\partial c}{\partial r}\right].$$

For Reynolds numbers above about 750 (based on the displacement thickness) r is greater than $1/\sqrt{2}$ and certain amplified modes will create a caustic along some path indicated in figure 3. Outside this caustic there are no periodic disturbances at all, not even damped ones, and the waves are thus contained within the limit

$$\frac{z}{t} < \frac{1}{2}\left[\alpha\frac{\partial c}{\partial\alpha} - r\frac{\partial c}{\partial r}\right]$$



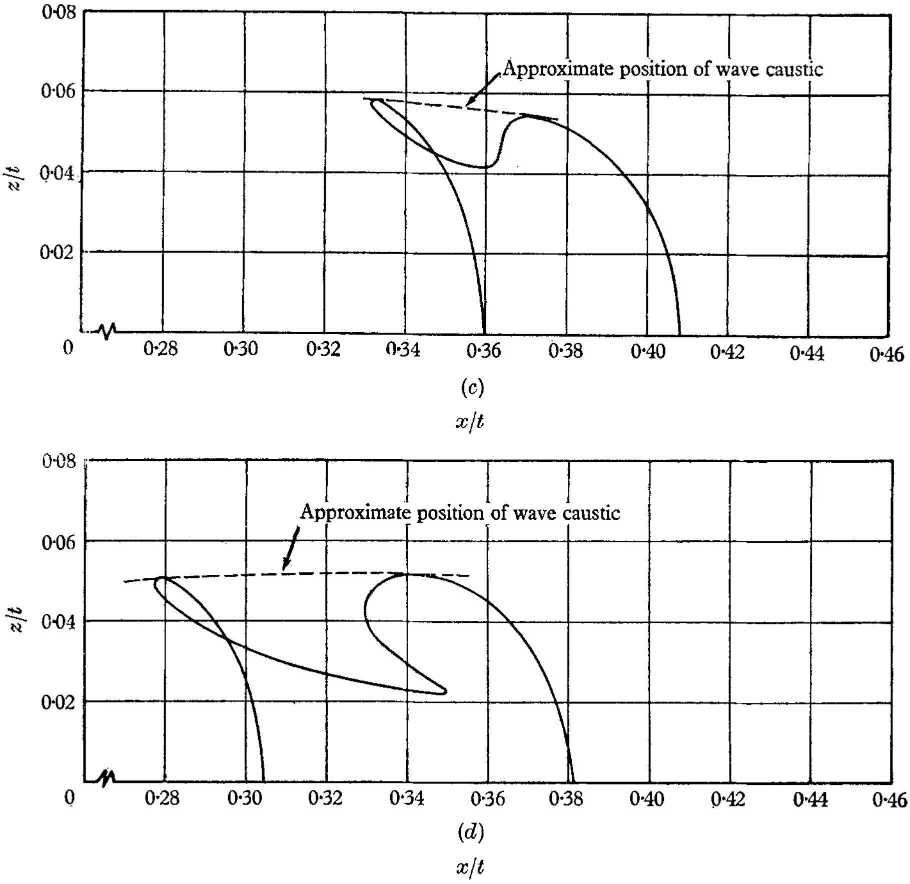


FIGURE 3. Wave envelopes in a flat-plate boundary layer at various Reynolds numbers. (*a*) $R_{\delta^*} = 750$; (*b*) $R_{\delta^*} = 1000$, (*c*) $R_{\delta^*} = 1500$, (*d*) $R_{\delta^*} = 3000$.

or approximately 0·06. Taking a typical value of x/t of about 0·4 in the area of the caustic we see that the maximum possible spread angle is around $\pm 10°$.

Far downstream where the exponential terms become dominant, significant disturbances only exist in the region bounded by the wave caustic and the neutral amplification boundary. However, at stations nearer the source, the actual amplitude distribution will have a region of large amplitude along the caustic outside this boundary. Although such disturbances ultimately decay in the linear theory presented here, the region around the caustic may be important as far as transition is concerned. Even at low Reynolds numbers below the critical when all waves are damped a caustic will occur and there is the possibility of strong non-linear interactions in such a region causing transition.

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